Exercises. Prove or disprove that the following functions are continuous (on their natural domain):

1. $f(x) = \sin(x)$;
2. $f(x) = k$, where $k \in \mathbb{R}$;
3. $f(x) = \sqrt{x}$;
4. $f(x) = |x|$;
5. $f(x) = \frac{1}{x}$;
6. $f(x) = \chi_{(0,\infty)}(x)$;
7. $f(x) = \chi_{\mathbb{Q}}(x)$;
8. $f(x) = x \cdot \chi_{\mathbb{Q}}(x)$.

Solutions. Note that the following proofs are written as rough work; you should rewrite these in an exam/assignment with your choice of δ at the top, before you rearrange $|f(x) - f(c)|$.

1. Let $\varepsilon > 0$. Then, if $|x - c| < \delta$, we have

$$\begin{aligned}|f(x) - f(c)| &= |\sin(x) - \sin(c)| \\ &= \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \\ &\leq \left|2\sin\left(\frac{x-c}{2}\right)\right| \\ &\leq \left|2\left(\frac{x-c}{2}\right)\right| \\ &= |x - c| \\ &< \delta\end{aligned}$$

Picking $\delta = \varepsilon$, we have

$$= \varepsilon$$

So this function is continuous. (In an exam, put "Choose $\delta = \varepsilon$" at the top, after "Let $\varepsilon > 0$", but you should use this kind of rough working to inform your choice of δ.)

2. Let $\varepsilon > 0$. Then,

$$\begin{aligned}|f(x) - f(c)| &= |k - k| \\ &= 0 \\ &< \varepsilon\end{aligned}$$

So this function is continuous. (Note that the value of δ is irrelevant here, so just put "Choose $\delta = 1$" at the top, or any other arbitrary positive value.)

3. Let $\varepsilon > 0$. Then, if $|x - c| < \delta$, we have

$$\begin{aligned}|f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\ &= \left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| \\ &= \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} \\ &< \frac{\delta}{|\sqrt{x}+\sqrt{c}|}\end{aligned}$$

Observe that $|\sqrt{x} + \sqrt{c}| \geq |\sqrt{c}|$, so

$$\leq \frac{\delta}{|\sqrt{c}|}$$

so if $\delta = \varepsilon\sqrt{c}$,

$$< \varepsilon$$

However, we divided by $\sqrt{c}$ in the above, so this proof is only valid for $c \neq 0$. For $c = 0$, whenever $|x - c| = |x| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= |\sqrt{x} - \sqrt{0}| \\ &= |\sqrt{x}| \\ &= \sqrt{|x|} \\ &< \sqrt{\delta} \end{aligned}$$

Picking $\delta = \varepsilon^2$, we have

$$= \varepsilon$$

which completes the proof.

4. Let $\varepsilon > 0$. Then, if $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= \big||x| - |c|\big| \\ &\leq |x - c| \\ &< \delta \end{aligned}$$

Picking $\delta = \varepsilon$, we have

$$= \varepsilon$$

5. Let $\varepsilon > 0$. Then, if $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\ &= \left|\frac{x - c}{xc}\right| \\ &< \frac{\delta}{|xc|} \end{aligned}$$

If we had $\delta = \varepsilon|xc|$, we would be done; however, δ cannot depend on x, so we aim to eliminate x from this expression.

If $\delta \leq \frac{|c|}{2}$, then

$$\begin{aligned} |x - c| &< \frac{|c|}{2} \\ \big||x| - |c|\big| &< \frac{|c|}{2} \end{aligned}$$

so by the interval property,

$$|c| - \frac{|c|}{2} < |x| < |c| + \frac{|c|}{2}$$

$$\frac{|c|}{2} < |x| < \frac{3|c|}{2}$$

Then,

$$\begin{aligned} |f(x) - f(c)| &< \frac{\delta}{|x||c|} \\ &\leq \frac{\delta}{\left|\frac{|c|}{2}\right| |c|} \\ &= \frac{2\delta}{|c|^2} \end{aligned}$$

Now, if $\delta \leq \frac{|c|^2}{2}$, we have

$$\leq \varepsilon$$

However, we earlier required that $|x - c| < \frac{|c|}{2}$, so we need δ to simultaneously satisfy $\delta \leq \frac{|c|}{2}$ and $\delta \leq \frac{|c|^2}{2}$. So let $\delta = \min\left(\frac{|c|}{2}, \frac{|c|^2}{2}\right)$.

6. Consider the sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{Q}$ defined by $x_n \coloneqq \frac{1}{n}$, converging to the point $c \coloneqq 0$. Then,

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} 1 \\ &= 1 \\ &\neq 0 \\ &= f(c) \end{aligned}$$

so $f = \chi_{(0,\infty)}$ is not (sequentially) continuous at 0. At every other point, f is constant and is continuous (proof similar to $Q1$).

7. Let $\varepsilon = \frac{1}{2}$, let $\delta > 0$, and recall that any interval in $\mathbb{R}$ of positive length contains both rational and irrational numbers.

   Let $c \in \mathbb{Q}$ and consider the interval $I_\delta = (c - \delta, c + \delta)$. There exists an irrational $x \in I_\delta$ as I_δ is has length $2\delta > 0$. By construction, $|x - c| < \delta$. Then,

$$\begin{aligned} |f(x) - f(c)| &= |1 - 0| \\ &= 1 \\ &\not< \frac{1}{2} \\ &= \varepsilon \end{aligned}$$

   The proof for the case $c \in \mathbb{R} \setminus \mathbb{Q}$ is symmetric.

8. Let $c = 0$, $\varepsilon > 0$, and $\delta = \varepsilon$, and suppose that $|x - c| = |x| < \delta$. If $x \in \mathbb{Q}$, then

$$\begin{aligned} |f(x) - f(c)| &= |f(x)| \\ &= |x| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

   If $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|f(x) - f(c)| = |f(x)|$$

$$= |0|$$
$$< \varepsilon$$

In either case, $|f(x) - f(c)| < \varepsilon$, so f is continuous at 0.

Let $c \in \mathbb{Q} \setminus \{0\}$, and consider the sequence defined by $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \mathbb{Q}$ defined by $x_n := c + \frac{\sqrt{2}}{n}$, that converges to the point c. Then,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} 0 \\ &= 0\end{aligned}$$

Then, we have $f(c) = c \neq 0$, so f is discontinuous at all points $c \in \mathbb{Q}$.

If $c \in \mathbb{R} \setminus \mathbb{Q}$, then instead consider the sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{Q}$ defined by $x_n := \frac{\lfloor c \cdot 10^n \rfloor}{10^n}$ (or alternatively, appeal to Example sheet 2, $Q1$ to non-constructively generate such a sequence), that converges to the point c. Then,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n \\ &= c \\ &\neq 0 \\ &= f(c)\end{aligned}$$

So f is discontinuous at all points $c \in \mathbb{R} \setminus \mathbb{Q}$.

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Example. Prove or disprove that the following series converge:

1. $\sum \frac{1}{n}$;
2. $\sum \frac{1}{n^2}$;
3. $\sum \frac{(-1)^n}{\sqrt{n}}$;
4. $\sum \frac{n^2}{n!}$;
5. $\sum \frac{\sin(n)}{n}$;
6. $\sum \frac{\sin(n)}{n^2}$;
7. $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$;
8. $\sum \frac{2^n+3^n}{5^n}$;
9. $\sum \frac{\cos(\pi n)}{n^2}$;
10. $\sum \frac{n^n}{(n!)^2}$.